On partitions of the unit interval generated by Brocot sequences.

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Let $p_{i,n}, i = 1, \dots, 2^{n-1}$ be the lengths of intervals between the neighboring fractions of Brocot sequence F_n. The asymptotic formula for $\sigma(F_n) = \sum_{i=1}^{N(n)} p_{i,n}^{\beta}$, improving known estimations, is obtained.

§1. Basic definitions and statements.

In the present work the partition of $[0,1]$ by the points of Brocot sequence is considered.

Brocot sequences F_n, $n = 1, 2, \dots$ are defined inductively in the following way. When $n = 1$ let $F_1 = \{0, 1\} = \{\frac{0}{1}, \frac{1}{1}\}$. Let $n \geq 1$ and for each $k \leq n$ sets F_k have been defined. Let's define F_{n+1}. Consider fractions from F_n, ordered by increase :

$$0 = x_{0,n} < x_{1,n} < \dots < x_{N(n),n} = 1, N(n) = 2^{n-1}. \tag{1}$$

Then

$$F_{n+1} = F_n \cup Q_{n+1},$$

where Q_{n+1} is the set of mediants of neighboring fractions in F_n, the given

$$Q_{n+1} = \{x_{i,n} \oplus x_{i-1,n}, i = 1, \dots, N(n)\},$$

where $\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}$. Elements in Q_n are known as Brocot fractions of order n. Brocot sequences (known also as Stern-Brocot sequences) appeared in [1], [2]. Main properties of Brocot sequences can be found in [3],pages 140-143.

Let us consider the partition of $[0,1]$ by fractions of F_n, that is with the given points like (1), let $p_{i,n} = x_{i,n} - x_{i-1,n}, i = 1, \dots, N(n)$ be lengths of $[x_{i-1,n}, x_{i,n})$. For fixed β we denote

$$\sigma_\beta(F_n) = \sum_{i=1}^{N(n)} p_{i,n}^{\beta}.$$

N. Moshchevitin and A.Zhigljavsky in [5] investigated the behavior of $\sigma(F_n)$ when n tends to infinity. The following asymptotic equality was proved there.

Theorem 1.

For any $\beta > 1$

$$\sigma_\beta(F_n) = \frac{2}{n^\beta}\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{\log(n)}{n^{(\beta+1)(2\beta-1)/(2\beta)}}\right), n \to \infty,$$

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where $\zeta(s)$ is Riemann ζ -function.
The main result of this work is proof of the following more precise theorem.
Theorem 2.
For any $\beta > 1$ holds

$$\sigma_\beta(F_n) = \frac{1}{n^\beta}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-2} C_k \frac{1}{n^{\beta+k}} + \sum_{0\le k<\beta-2} {C^*}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{\log^{3\beta} n}{n^{3\beta-2}}\right),$$

where $C_k(\beta), 1 \le k \le 2\beta-2$, ${C^*}_k(\beta), 0 \le k \le \beta-2$ are positive constants, depending on β.
When $\beta \in (1, 1.5]$ the formula in the theorem 2 is

$$\sigma_\beta(F_n) = \frac{1}{n^\beta}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{\log^{3\beta} n}{n^{3\beta-2}}\right).$$

The error term here is better than in theorem 1, because when $1 < \beta \le 1.5$ we have $3\beta - 2 > \frac{(\beta+1)(2\beta-1)}{(2\beta)}$.
When $\beta > 1.5$ theorem 2 gives the additional terms in asymptotic.
Note, that the history of the problem and review of some results is presented in the introduction of [5].

§2. Some notation and formulation of auxiliary result.

It's well known, that the sum of partial quotients in the continued fraction representation for Brocot fractions of order n equals n, i. e.

$$Q_n = \{\frac{p}{q} = [a_1, \dots, a_t], a_t \ge 2, a_1 + \dots + a_t = n\}.$$

Let A be the set of all integer vectors $a = (a_1, \dots, a_t), t \ge 1, a_j \ge 1, j = 1, \dots, t-1$ and $a_t \ge 2$.
Let

$$A_n = \{a = (a_1, \dots, a_t) \in A | a_1 + \dots + a_t = n\}.$$

Each $a = (a_1, \dots, a_t) \in A$ is associated with the continued fraction $[0; a_1, \dots, a_t]$ (as integer part always equals zero, we simply denote it as $[a_1, \dots, a_t]$) and corresponding continuant $\langle a_1, \dots, a_t\rangle$, empty continuant equals 1, -1 continuant equals 0. By construction, for any $n > 1$ each fraction in $F_n \backslash (F_1 \cup Q_n)$ has two neighbors in Q_n, and each fraction $\frac{p}{q} \in Q_n$ has two neighbors $\frac{p_-}{q_-}$ and $\frac{p_+}{q_+}$ in $F_n \backslash Q_n$.

Lemma 1.
For each $a \in A_n$, the fraction $\frac{p}{q} \in Q_n$ with denominator equal to continuant $q = \langle a_1, \dots, a_t\rangle$ has two neighbors in F_n with denominators, equal to continuants $q_- = \langle a_1, \dots, a_{t-1}\rangle$ and $q_+ = \langle a_1, \dots, a_t - 1\rangle$. Similarly, any fraction $\frac{p}{q} \in F_{n-1} \backslash F_1$ with denominator equal to continuant $\langle a_1, \dots, a_t\rangle$ has two neighbors in F_n with denominators, equal to continuants $\langle a_1, \dots, a_t, n - (a_1 + \dots + a_t)\rangle$

and $\langle a_1, \ldots, a_t - 1, 1, n - (a_1 + \cdots + a_t)\rangle$.
Proof is a simple induction with respect to n (see. [5]).

To prove theorem 2 we need the following auxiliary result, that can be of self-contained interest.
Let

$$\sigma_\beta(n) = \sum_{(a_1,\ldots,a_t)\in A_n} \frac{1}{\langle a_1, \ldots, a_t\rangle^{2\beta}}$$

with the fixed $\beta > 1$.
Theorem 3.
For each $\beta > 1$ with some positive constants C'_k, depending on β, holds

$$\sigma_\beta(n) = \frac{1}{n^{2\beta}} \left(\frac{\zeta\,(2\beta-1)}{\zeta\,(2\beta)} + 2\left(\frac{\zeta\,(2\beta-1)}{\zeta\,(2\beta)}\right)^2 \right) + \sum_{1\le k<2\beta-2} C'_k \frac{1}{n^{2\beta+k}} + O\left(\frac{\log^{4\beta} n}{n^{4\beta-2}}\right),$$

In fact, in order to prove theorem 2 it is sufficient to obtain the main term in asymptotic in theorem 3. This main term will be obtained in lemma 9 further. Note, that lemma 9 is the weaker variant of theorem 3 and it's actually used to prove theorem 3 in all completeness.

§3. Auxiliary statements.

Proof of theorem 3 uses splitting of σ which is the sum over A_n into the sums over smaller subsets of indices.
Let r and w be some integers, satisfying the conditions $r \ge 1$ and $1 \le w \le n$. Then $A_n = A^{(1)}_{n,1} \sqcup A^{(1)}_{n,2}$, where

$$A^{(1)}_{n,1} = \{a = (a_1, \ldots, a_t) \in A_n \mid \langle a_1, \ldots, a_t\rangle < n^r\}$$

$$A^{(1)}_{n,2} = A_n \backslash A^{(1)}_{n,1} = \{a \in A_n \mid \langle a_1, \ldots, a_t\rangle \ge n^r\}.$$

Then split $A^{(1)}_{n,1}$ into

$$A^{(2)}_{n,1} = \{a \in A^{(1)}_{n,1} \mid \max_{1\le j\le t} a_j > n - w\}$$

and

$$A^{(2)}_{n,2} = A^{(1)}_{n,1} \backslash A^{(2)}_{n,1} = \{a \in A^{(1)}_{n,1} \mid \max_{1\le j\le t} a_j \le n - w\}.$$

Thus, all $a \in A^{(2)}_{n,1}$ has at least one very large partial quotient; on the other hand, all a_j for $a \in A^{(2)}_{n,2}$ are relatively small.

So, $A_n = A^{(2)}_{n,1} \sqcup A^{(2)}_{n,2} \sqcup A^{(1)}_{n,2}$.

For subset $A_{n,i}^{(j)}$ of A denote

$$\Sigma_{n,i}^{(j)} = \sum_{a \in A_{n,i}^{(j)}} \frac{1}{q^{2\beta}}.$$

Thus,

$$\sigma_\beta(n) = \Sigma_{n,2}^{(1)} + \Sigma_{n,1}^{(2)} + \Sigma_{n,2}^{(2)}.$$

Let us estimate these sums separately.

Lemma 2.

Let $n \geq 2, a = (a_1, \ldots, a_t) \in A_n$. We have $q = q_+ + q_- \leq nq_-$, $q_- \leq q_+ \leq a_t q_-$,

$$\sum_{a \in A_n} \left(\frac{1}{qq_-} + \frac{1}{qq_+} \right) = 1.$$

The proof of this lemma is given in [3].

Lemma 3.

For each $n \geq 1$ holds

$$\Sigma_{n,2}^{(1)} \leq 2^{\beta-1} \frac{1}{n^{2r(\beta-1)}}. \tag{2}$$

(Lemma 2 is similar to lemma 3 from [5].)

Proof.

As $q\,(a) = \langle a_1, \ldots, a_t \rangle \geq n^r$ for each a in $A_{n,2}^{(1)}$, then using lemma 2, we obtain

$$\Sigma_{n,2}^{(1)} \leq \sum_{A_{n,2}^{(1)}} \frac{1}{(qq_+)^\beta} \leq \max_{A_{n,2}^{(1)}} \frac{1}{(qq_+)^{\beta-1}} \sum_{A_{n,2}^{(1)}} \frac{1}{qq_+} \leq$$

$$\leq \max_{A_{n,2}^{(1)}} \frac{1}{(qq_+)^{\beta-1}} \sum_{A_{n,2}^{(1)}} \left(\frac{1}{qq_+} + \frac{1}{qq_-} \right) \leq \max_{A_{n,2}^{(1)}} \frac{1}{(qq_+)^{\beta-1}} \leq$$

$$\leq \frac{2^{\beta-1}}{q^{2(\beta-1)}} \leq \frac{2^{\beta-1}}{n^{2r(\beta-1)}}.$$

Lemma is proved.

Lemma 4.

For each $a \in A_{n,1}^{(1)}$ when $n \geq 2$ holds

$$t \leq Kr \log n, K = \left(\log \frac{\sqrt{5}+1}{2} \right)^{-1}.$$

Proof.

For each $a \in A_{n,1}^{(1)}$ holds

$$\left(\frac{\sqrt{5}+1}{2} \right)^t \leq \langle a_1, \ldots, a_t \rangle \leq n^r, \quad t \left(\log \frac{\sqrt{5}+1}{2} \right) \leq r \log n.$$

Lemma is proved.

Lemma 5.
When $n \to \infty$, the following estimate for $\sum_{n,2}^{(2)}$ holds:

$$\Sigma_{n,2}^{(2)} \ll \frac{n^2 \log^{4\beta} n}{w^{4\beta}}. \tag{3}$$

Note, that lemma 5 can be improved, but it won't effect the main result.
Proof.
According to lemma 4 for each $a \in A_{n,1}^{(1)}$ holds $t \le Kr \log n$. As $n = a_1 + \ldots + a_t \le t \max a_j$, then $\max a_j \ge \frac{n}{Kr \log n}$.
Let $a \in A_{n,2}^{(2)}$ and j be such that $a_j = \max\{a_1, \ldots, a_t\}$. As $a_j \le n - w$, then for the sum of the rest a_j we have $\sum\limits_{i \ne j} a_j \ge w$, and, similarly to the above, $\max_{i \ne j} a_j \ge \frac{w}{Kr \log n}$.
This implies, that for each $a \in A_{n,2}^{(2)}$ there exist 2 different partial quotients a_k and $a_l, k \ne l$, that $a_k \ge \frac{w}{Kr \log n}$, $a_l \ge \frac{w}{Kr \log n}$. Hence,

$$\Sigma_{n,2}^{(2)} \le \sum_{\substack{a_1 + \ldots + a_t = n, \\ \langle a_1, \ldots, a_t \rangle \le n^r, \\ \exists k, l, k \ne l : a_k, a_l \ge \frac{w}{K \log n}}} \frac{1}{q^{2\beta}}.$$

Using the well known formula for continuants (see. [3]), we get

$$\begin{aligned} \langle a_1, \ldots, a_i, \ldots, a_t \rangle &= a_i \langle a_1, \ldots, a_{i-1} \rangle \langle a_{i+1}, \ldots, a_t \rangle \times \\ &\times \left(1 + \frac{1}{a_i}[a_{i-1}, \ldots, a_1] + \frac{1}{a_i}[a_{i+1}, \ldots, a_t] \right) = \\ = a_i \langle a_1, \ldots, a_{i-1} \rangle \langle a_{i+1}, \ldots, a_t \rangle &\left(1 + \frac{1}{a_i}[a_{i-1}, \ldots, a_1] \right) \times \\ &\times \left(1 + [a_{i+1}, \ldots a_t] \frac{1}{a_i + [a_{i-1}, \ldots, a_1]} \right), \end{aligned} \tag{4}$$

Therefore,

$$q\,(a) \ge a_k a_l \langle a_1, .., a_{\min(k,l)-1} \rangle \langle a_{\min(k,l)+1}, .., a_{\max(k,l)-1} \rangle \langle a_{\max(k,l)+1}, .., a_t \rangle.$$

Note, that with the X fixed the elements of set

$$\{a \in A_n | \langle a_1, \ldots, a_t \rangle \le n^r, \exists k, l, k \ne l : a_k, a_l \ge X\}$$

look like

$$\left(a_1, \ldots a_{\min(k,l)-1}, T, a_{\min(k,l)+1}, \ldots, a_{\max(k,l)-1}, P, a_{\max(k,l)+1}, \ldots, a_t\right),$$

where $T, P \geq X$, lengths of

$$\left(a_1, \ldots, a_{\min(k,l)-1}\right),$$

$$\left(a_{\min(k,l)+1}, \ldots, a_{\max(k,l)-1}\right)$$

and

$$\left(a_{\max(k,l)+1}, \ldots, a_t\right)$$

are not fixed, and the sum is

$$a_1 + \cdots + a_{l-1} + a_{l+1} + \cdots + a_{k-1} + a_{k+1} + \cdots + a_t = n - T - P.$$

Let

$$a_1 + \cdots + a_{\min(k,l)-1} = u,$$

$$a_{\min(k,l)+1} + \cdots + a_{\max(k,l)-1} = v,$$

$$a_{\max(k,l)+1} + \cdots + a_t = s,$$

$$u + v + s = n - T - P.$$

Thus,

$$\Sigma_{n,2}^{(2)} \leq \sum_{\substack{a_1 + \ldots + a_t = n, \\ \langle a_1, \ldots, a_t \rangle \leq n^r, \\ \exists k, l, k \neq l : a_k, a_l \geq \frac{w}{K \log n}}} \frac{1}{(a_k a_l)^{2\beta}} \times$$

$$\times \frac{1}{\left(\langle a_1, .., a_{\min(k,l)-1} \rangle \langle a_{\min(k,l)+1}, .., a_{\max(k,l)-1} \rangle \langle a_{\max(k,l)+1}, .., a_t \rangle\right)^{2\beta}} \leq$$

$$\leq \sum_{\substack{a_k + a_l \leq n, \\ a_k, a_l \geq \frac{w}{K \log n}}} \frac{1}{(a_k a_l)^{2\beta}} \times$$

$$\times \sum_{u+v+s=n-a_k-a_l} \sum_{a_1+\cdots+a_{\min(k,l)-1}=u} \frac{1}{\langle a_1, \ldots, a_{\min(k,l)-1} \rangle^{2\beta}} \times$$

$$\times \sum_{a_{\min(k,l)+1}+\cdots+a_{\max(k,l)-1}=v} \frac{1}{\langle a_{\min(k,l)+1}, \ldots, a_{\max(k,l)-1} \rangle^{2\beta}} \times$$

$$\times \sum_{a_{\max(k,l)+1}+\cdots+a_t=s} \frac{1}{\langle a_{\max(k,l)+1}, \ldots, a_t \rangle^{2\beta}}.$$

Let's estimate the internal sum.

$$\sum_{u+v+s=n-a_k-a_l} \sum_{a_1+\cdots+a_{\min(k,l)-1}=u} \frac{1}{\langle a_1, \ldots, a_{\min(k,l)-1} \rangle^{2\beta}}$$

$$\sum_{a_{\min(k,l)+1}+\cdots+a_{\max(k,l)-1}=v} \frac{1}{\langle a_{\min(k,l)+1},\ldots,a_{\max(k,l)-1}\rangle^{2\beta}} \sum_{a_{\max(k,l)+1}+\cdots+a_t=s} \frac{1}{\langle a_{\max(k,l)+1},\ldots,a_t\rangle^{2\beta}} =$$

$$= \sum_{u+v+s=n-a_k-a_l} \sum_{x_1+\cdots+x_r=u} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \sum_{z_1+\cdots+z_g=s} \frac{1}{\langle z_1,\ldots,z_g\rangle^{2\beta}} \leq$$

$$\leq \sum_{u+v+s\leq n} \sum_{x_1+\cdots+x_r=u} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \sum_{z_1+\cdots+z_g=s} \frac{1}{\langle z_1,\ldots,z_g\rangle^{2\beta}} \leq$$

$$\leq \sum_{u+v+s\leq \infty} \sum_{x_1+\cdots+x_r=u} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \sum_{z_1+\cdots+z_g=s} \frac{1}{\langle z_1,\ldots,z_g\rangle^{2\beta}} =$$

$$= \sum_{x_1+\cdots+x_r\leq\infty} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h\leq\infty} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \sum_{z_1+\cdots+z_g\leq\infty} \frac{1}{\langle z_1,\ldots,z_g\rangle^{2\beta}} \leq$$

$$\leq \left(\sum_{x_1+\cdots+x_r\leq\infty} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \right)^3 \leq \left(1 + 2 \sum_{\substack{x_1+\cdots+x_r\leq\infty,\\ x_r\geq 2}} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \right)^3 =$$

$$= \left(1 + 2\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right)^3.$$

Then for the external sum the following estimation holds:

$$\sum_{\substack{a_k+a_l\leq n,\\ a_k,a_l\geq \frac{w}{K\log n}}} \frac{1}{(a_k a_l)^{2\beta}} \ll \left(n - 2\frac{w}{K\log n}\right)^2 \left(\frac{K^{4\beta}\log^{4\beta} n}{w^{4\beta}}\right),$$

where $\left(n - 2\frac{w}{K\log n}\right)^2$ is the number of items in the sum, and $\frac{K^{4\beta}\log^{4\beta} n}{w^{4\beta}}$ is the estimate for $\frac{1}{(a_k a_l)^{2\beta}}$. Thus, we obtain that

$$\Sigma_{n,2}^{(2)} \ll \left(n - 2\frac{w}{K\log n}\right)^2 \left(\frac{\log^{4\beta} n}{w^{4\beta}}\right) = O\left(\frac{n^2\log^{4\beta} n}{w^{4\beta}}\right).$$

Lemma is proved.

Lemma 6.

When $w < \frac{n}{2}$

$$Q = \{a \in A_n | \exists j : a_j > n - w\} = \bigsqcup_{X=n-w}^{n} \bigsqcup_{u+v=n-X} P(u,v,X),$$

where

$$P(u,v,X) = \{a \in A | a = (a_1,\ldots,a_t,X,a'_1,\ldots,a'_{t'}),$$

$$a_1 + \cdots + a_t = u, {a'}_1 + \cdots + {a'}_{t'} = v\},$$

and symbol $\bigsqcup$ *means, that sets* $P(u, v, X)$ *and* $P(u', v', X')$ *don't intersect, when* $(u, v, X) \neq (u', v', X')$.

Proof.

Let $a \in Q$, then there exists the partial quotient $a_j = Y > n - w$, hence this element is in $P(a_1 + \cdots + a_{j-1}, a_{j+1} + \cdots + a_t, Y)$.

Inversely, if $a \in \bigsqcup_{X=w}^{n} \bigsqcup_{u+v=n-X} P(u, v, X)$, then $a \in A_n$ and there exists the partial quotient $a_i > n - w$.

Let's prove that element from Q can't belong to several sets $P(u, v, X)$ at the same time. If it's not true and there exists $a \in Q$, such that $a \in P(u, v, X)$ and $a \in P(u^*, v^*, X^*)$, then it can be represented as

$$a = (a_1, \ldots, a_{i-1}, X, a_{i+1}, \ldots, a_t),$$

$$a_1 + \cdots + a_{i-1} = u, a_{i+1} + \cdots + a_t = v, u + v = n - X$$

and

$$a = \left(a_1^*, \ldots, a_{j-1}^*, X^*, a_{j+1}^*, \ldots, a_t^*\right),$$

$$a_1^* + \cdots + a_{j-1}^* = u^*, a_{j+1}^* + \cdots + a_t^* = v^*, u^* + v^* = n - X^*.$$

Let $i \neq j$. Then in a there exist two partial quotients, larger than $\frac{n}{2}$, and hence $\sum a_i > n$, that contradicts the fact that $a \in Q$.

Hence, $i = j$, i. e. $X = X^*$, and, obviously, $(u, v) = (u^*, v^*)$, the given sets $P(u, v, X)$ and $P(u^*, v^*, X^*)$ are the same.

Lemma is proved.

Lemma 7. *Let* $w < \frac{n}{2}$.

Then for the sum

$$R_0 = \sum_{\substack{a \in A_n, \\ \exists j : a_j > n - w}} \frac{1}{\langle a_1, \ldots, a_{j-1} \rangle^{2\beta} \langle a_{j+1}, \ldots, a_t \rangle^{2\beta}}$$

the following asymptotic formula holds:

$$R_0 = \sum_{\substack{a \in A_n, \\ \exists j : a_j > n - w}} \frac{1}{\langle a_1, \ldots, a_{j-1} \rangle^{2\beta} \langle a_{j+1}, \ldots, a_t \rangle^{2\beta}} = C_0 + O\left(\frac{1}{w^{2(\beta-1)}}\right),$$

where

$$C_0 = \frac{\zeta(2\beta - 1)}{\zeta(2\beta)} + 2\left(\frac{\zeta(2\beta - 1)}{\zeta(2\beta)}\right)^2.$$

Proof.

According to lemma 6,

$$R_0 = \sum_{\substack{a \in A_n, \\ \exists j : a_j > n - w}} \frac{1}{\langle a_1, \ldots, a_{j-1} \rangle^{2\beta} \langle a_{j+1}, \ldots, a_t \rangle^{2\beta}} =$$

$$= \sum_{X=n-w}^{n} \sum_{u+v=n-X} \sum_{\substack{a_1+\cdots+a_{j-1}=u,\\ a_{j+1}+\cdots+a_t=v}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}} =$$

$$\sum_{u+v\le w} \sum_{a_1+\cdots+a_{j-1}=u} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}} \sum_{\substack{a_{j+1}+\cdots+a_t=v,\\ a_t\ge 2}} \frac{1}{\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}.$$

Let's split the sum into 2 parts, separating item $\sum_1$ with $u=1$. Replacing sum with $u>1$ to the doubled $\sum_2$ over the set of indices with the last partial quotient larger or equal to 2, i.e. over A_n, we obtain

$$R_0 = \Sigma_1 + 2\Sigma_2,$$

where

$$\Sigma_1 = \sum_{v\le w-1} \sum_{\substack{a_{j+1}+\cdots+a_t=v,\\ a_t\ge 2}} \frac{1}{\langle a_{j+1},\ldots,a_t\rangle^{2\beta}} = \sum_{v\le w-1} \sigma_\beta(v),$$

$$\Sigma_2 = \sum_{u+v\le w} \sum_{\substack{a_1+\cdots+a_{j-1}=u,\\ a_{j-1}\ge 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}} \times$$

$$\times \sum_{\substack{a_{j+1}+\cdots+a_t=v,\\ a_t\ge 2}} \frac{1}{\langle a_{j+1},\ldots,a_t\rangle^{2\beta}} =$$

$$= \sum_{u+v\le w} \sigma_\beta(u)\sigma_\beta(v).$$

For Σ_1, doing the calculations like in the proof of lemma 7 of the theorem 1 (see [3]), we get

$$\Sigma_1 = \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{1}{w^{2\beta-2}}\right).$$

Let's get the asymptotic formula for Σ_2 :

$$\Sigma_2 \le \sum_{\substack{a_1+..+a_k\le\infty,\\ a_k\ge 2}} \frac{1}{\langle a_1,\ldots,a_k\rangle^{2\beta}} \sum_{\substack{a_1+..+a_l\le\infty,\\ a_l\ge 2}} \frac{1}{\langle a_1,\ldots,a_l\rangle^{2\beta}} =$$

$$= \left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right)^2.$$

On the other hand,

$$\Sigma_2 \geq \sum_{\substack{a_1+..+a_k \leq \frac{w}{2},\\ a_k \geq 2}} \frac{1}{\langle a_1,\ldots,a_k\rangle^{2\beta}} \sum_{\substack{a_1+..+a_l \leq \frac{w}{2},\\ a_l \geq 2}} \frac{1}{\langle a_1,\ldots,a_l\rangle^{2\beta}},$$

because all these items are in $\sum_2$.

$$\Sigma_2 \geq \sum_{\substack{a_1+..+a_k \leq \infty,\\ a_k \geq 2}} \frac{1}{\langle a_1,\ldots,a_k\rangle^{2\beta}} \sum_{\substack{a_1+..+a_l \leq \infty,\\ a_l \geq 2}} \frac{1}{\langle a_1,\ldots,a_l\rangle^{2\beta}} -$$

$$-O\left(\frac{1}{w^{2\beta-2}}\right) =$$

$$= \left(\frac{\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)}\right)^2 + O\left(\frac{1}{w^{2(\beta-1)}}\right),$$

i.e.

$$\Sigma_2 = \left(\frac{\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)}\right)^2 + O\left(\frac{1}{w^{2(\beta-1)}}\right).$$

Thus,

$$R_0 = C_0 + O\left(\frac{1}{w^{2(\beta-1)}}\right).$$

Lemma is proved.
Lemma 8. *When* $w \leq \frac{n}{2} - 2$

$$\Sigma_{n,1}^{(2)} = \frac{C_0}{n^{2\beta}} + O\left(\frac{w}{n^{2\beta+1}}\right) + O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}}\right) + O\left(\frac{1}{n^{2r(\beta-1)}}\right). \qquad (5)$$

where C_0 *is defined in Lemma 7.*

Proof.
Note that when $w \leq \frac{n}{2} - 2$, each element of $A_{n,1}^{(2)}$ has the only one partial quotient, that is larger, than $n - w$.

$$\Sigma_{n,1}^{(2)} = \sum_{\substack{a \in A_n,\\ q(a) < n^r,\\ \exists i : a_i > n-w}} \frac{1}{q^{2\beta}} =$$

$$= \sum_{\substack{a \in A_n,\\ \exists i : a_i > n-w}} \frac{1}{q^{2\beta}} - \sum_{\substack{a \in A_n,\\ q(a) \geq n^r,\\ \exists i : a_i > n-w}} \frac{1}{q^{2\beta}}.$$

The second sum is estimated according to Lemma 3.

$$\sum_{\substack{a \in A_n, \\ q(a) \geq n^r, \\ \exists i : a_i > n-w}} \frac{1}{q^{2\beta}} = O\left(\frac{1}{n^{2r(\beta-1)}}\right)$$

Let's estimate the first sum. Let $a_i = n + O(w)$.
Using the formula for continuants (4), obtain

$$q(a) = (a_i + [a_{i-1}, \ldots, a_1] + [a_{i+1}, \ldots, a_t]) \langle a_1, \ldots, a_{i-1} \rangle \langle a_{i+1}, \ldots, a_t \rangle =$$

$$= n\left(1 + \frac{w}{n} \cdot \theta\right) \langle a_1, \ldots, a_{i-1} \rangle \langle a_{i+1}, \ldots, a_t \rangle,$$

where $\theta = \theta(w, n), |\theta| \leq 1$. Then, considering $\frac{1}{q(a)^{2\beta}}$ as the function of argument a_i and expanding it in Taylor series according to argument $\frac{w}{n} \cdot \theta$, we obtain

$$\frac{1}{q(a)^{2\beta}} = \frac{1}{n^{2\beta} \langle a_1, \ldots, a_{j-1} \rangle^{2\beta} \langle a_{j+1}, \ldots, a_t \rangle^{2\beta}} \left(1 + O\left(\frac{w}{n}\right)\right).$$

Thus,

$$\sum_{\substack{a \in A_n, \\ \exists j : a_j > n-w}} \frac{1}{q^{2\beta}} =$$

$$= \sum_{\substack{a \in A_n, \\ \exists j : a_j > n-w}} \frac{1}{n^{2\beta} \langle a_1, \ldots, a_{j-1} \rangle^{2\beta} \langle a_{j+1}, \ldots, a_t \rangle^{2\beta}} \left(1 + O\left(\frac{w}{n}\right)\right) =$$

$$= \frac{1}{n^{2\beta}} \sum_{\substack{a \in A_n, \\ \exists j : a_j > n-w}} \frac{1}{\langle a_1, \ldots, a_{j-1} \rangle^{2\beta} \langle a_{j+1}, \ldots, a_t \rangle^{2\beta}} \left(1 + O\left(\frac{w}{n}\right)\right).$$

Hence, according to lemma 7

$$\sum_{\substack{a \in A_n, \\ \exists i : a_i > n-w}} \frac{1}{q^{2\beta}} = \frac{1}{n^{2\beta}} \left(C_0 + O\left(\frac{1}{w^{2(\beta-1)}}\right)\right) \left(1 + O\left(\frac{w}{n}\right)\right) =$$

$$= \frac{C_0}{n^{2\beta}} + O\left(\frac{w}{n^{2\beta+1}}\right) + O\left(\frac{1}{n^{2\beta} w^{2(\beta-1)}}\right).$$

Lemma is proved.
The following lemma is the weaker variant of theorem 3, which will be used to

prove more precise result.
Lemma 9. *When $\beta > 1$ we get*

$$\sigma_\beta(n) = \frac{C_0}{n^{2\beta}} + O\left(\frac{\log^{\frac{4\beta}{4\beta+1}} n}{n^{2\beta+1-\frac{2\beta+3}{4\beta+1}}}\right),$$

Proof.
Using (2), (3), (5), we get

$$\sigma_\beta(n) = \Sigma_{n,2}^{(1)} + \Sigma_{n,1}^{(2)} + \Sigma_{n,2}^{(2)} = \frac{C_0}{n^{2\beta}} + O\left(\frac{1}{n^{2\beta} w^{2(\beta-1)}}\right) + O\left(\frac{w}{n^{2\beta+1}}\right) +$$

$$+O\left(\frac{1}{n^{2r(\beta-1)}}\right) + O\left(\frac{n^2 \log^{4\beta} n}{w^{4\beta}}\right).$$

Optimizing according to w and r

$$w = \min\left\{\frac{n}{2} - 2, n^{\frac{2\beta+3}{4\beta+1}} \log^{\frac{4\beta}{4\beta+1}} n\right\},$$

$$r = \frac{2\beta+1}{2(\beta-1)},$$

we obtain

$$\sigma(F_n) = \frac{C_0}{n^{2\beta}} + O\left(\frac{\log^{\frac{4\beta}{4\beta+1}} n}{n^{2\beta+1-\frac{2\beta+3}{4\beta+1}}}\right).$$

Lemma is proved.

§4. Main lemma and final step of proving theorem 3.

Let's consider the sum $\Sigma_{n,1}^{(2)}$.
Lemma 10. *Let $w < \frac{n}{2}$.*
In case 2β is integer,

$$\Sigma_{n,1}^{(2)} = \frac{C_0}{n^{2\beta}} + \\ + \sum_{1\le k<2\beta-2} {C'}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta} w^{2(\beta-1)}} + \frac{\log n}{n^{4\beta-2}} + \frac{1}{n^{2r(\beta-1)}}\right), \tag{6}$$

in other case,

$$\Sigma_{n,1}^{(2)} = \frac{C_0}{n^{2\beta}} + \\ + \sum_{1\le k<2\beta-2} {C'}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta} w^{2(\beta-1)}} + \frac{1}{n^{2r(\beta-1)}}\right), \tag{7}$$

where C'_k are some constants.

Final step of proving theorem 3. Using (2), (3), (6) and (7) error term R in case 2β is integer equals

$$R = O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{\log n}{n^{4\beta-2}} + \frac{1}{n^{2r(\beta-1)}} + \frac{n^2\log^{4\beta} n}{w^{4\beta}}\right),$$

in case 2β is not integer equals

$$R = O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{1}{n^{2r(\beta-1)}} + \frac{n^2\log^{4\beta} n}{w^{4\beta}}\right).$$

Let $w = \frac{n}{2} - 2$, $r = \frac{2\beta-1}{\beta-1}$. Substituting the value of w and r, we get

$$R = O\left(\frac{\log^{4\beta} n}{n^{4\beta-2}}\right),$$

so theorem 3 follows.

Proof of lemma 10.

$$\Sigma_{n,1}^{(2)} = \sum_{\substack{a\in A_n,\\ q(a)<n^r,\\ \exists i: a_i>n-w}} \frac{1}{q^{2\beta}} = \sum_{\substack{a\in A_n,\\ \exists i: a_i>n-w}} \frac{1}{q^{2\beta}} - \sum_{\substack{a\in A_n,\\ q(a)\ge n^r,\\ \exists i: a_i>n-w}} \frac{1}{q^{2\beta}} \tag{8}$$

Second sum in (8) can be estimated according to lemma 3 as

$$\sum_{\substack{a\in A_n,\\ q(a)\ge n^r,\\ \exists i: a_i>n-w}} \frac{1}{q^{2\beta}} = O\left(\frac{1}{n^{2r(\beta-1)}}\right).$$

Let $a_i = n - v$, where $v = 1, \ldots, w-1$. Using (4), we obtain

$$q(a) = \langle a_1, \ldots, a_{i-1}\rangle\langle a_{i+1}, \ldots, a_t\rangle \left(a_i + [a_{i-1}, \ldots, a_1] + [a_{i+1}, \ldots a_t]\right) =$$

$$= n\langle a_1, \ldots, a_{i-1}\rangle\langle a_{i+1}, \ldots, a_t\rangle \left(1 - \frac{v}{n} + \frac{1}{n}\left([a_{i-1}, \ldots, a_1] + [a_{i+1}, \ldots a_t]\right)\right).$$

Then, expanding into Taylor series according to

$$\frac{v}{n} - \frac{1}{n}\left([a_{i-1}, \ldots, a_1] + [a_{i+1}, \ldots a_t]\right)$$

function

$$\frac{1}{q\left(a\right)^{2\beta}},$$

when

$$|\frac{v}{n}-\frac{1}{n}\left(\left[a_{i-1},\ldots,a_1\right]+\left[a_{i+1},\ldots a_t\right]\right)|<1$$

we get absolutely converging series

$$\frac{1}{q(a)^{2\beta}}=\frac{1}{n^{2\beta}\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}\cdot$$

$$\cdot\left(1+\sum_{k=1}^{\infty}\frac{2\beta\cdots(2\beta+k-1)}{k!}\left(\frac{v}{n}-\frac{1}{n}(\left[a_{i-1},\ldots,a_1\right]+\left[a_{i+1},\ldots a_t\right])\right)^k\right)=$$

$$=\frac{1}{n^{2\beta}\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}+$$

$$+\frac{1}{n^{2\beta}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}\sum_{k=1}^{\infty}\frac{1}{n^k}\gamma_k(2\beta)\left(v-(\left[a_{i-1},\ldots,a_1\right]+\left[a_{i+1},\ldots a_t\right])\right)^k, \tag{9}$$

where

$$\gamma_k(\beta)=\frac{\beta\cdots(\beta+k-1)}{k!}. \tag{10}$$

After substituting (9) to (8) regarding to lemma 6 with the given w, replacing sum according to a_i with sum according to v, we get

$$\sum_{\substack{a\in A_n,\\ \exists j:a_j>n-w}}\frac{1}{q^{2\beta}}=\sum_{v=1}^{w-1}\sum_{u+s=v}\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}=u,\\ a_{j+1}+\cdots+a_t=s}}\frac{1}{n^{2\beta}\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}+$$

$$+\frac{1}{n^{2\beta}}\sum_{v=1}^{w-1}\sum_{u+s=v}\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}=u,\\ a_{j+1}+\cdots+a_t=s}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}\sum_{k=1}^{\infty}\frac{1}{n^k}\gamma_k(2\beta)\cdot$$

$$\cdot\left(v-(\left[a_{i-1},\ldots,a_1\right]+\left[a_{i+1},\ldots a_t\right])\right)^k. \tag{11}$$

Let's consider the main term in asymptotic formula we've got. According to lemma 7,

$$\frac{1}{n^{2\beta}}\sum_{u+v\le w}\sum_{a_1+..+a_k=u}\frac{1}{\langle a_1,\ldots,a_k\rangle^{2\beta}}\sum_{a_1+..+a_l=v,a_l\ge 2}\frac{1}{\langle a_1,\ldots,a_l\rangle^{2\beta}}=$$

$$=\frac{C_0}{n^{2\beta}}+O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}}\right).$$

Now let's consider the error term:

$$\frac{1}{n^{2\beta}}\sum_{k=1}^{\infty}\frac{1}{n^k}\cdot\gamma_k(2\beta)\sum_{v=1}^{w-1}\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}+a_{j+1}+\cdots+a_t=v}}$$

$$\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots a_t])\right)^k+O\left(\frac{1}{n^{2r(\beta-1)}}\right)=$$

$$=\frac{1}{n^{2\beta}}\sum_{k=1}^{\infty}\frac{R_k}{n^k}+O\left(\frac{1}{n^{2r(\beta-1)}}\right).$$

Coefficient R_k at kth term is equal to

$$R_k=\gamma_k(2\beta)\cdot\sum_{v=1}^{w-1}\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}+a_{j+1}+\cdots+a_t=v}}$$

$$\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots a_t])\right)^k=$$

$$=\left(\sum_{v=1}^{\infty}-\sum_{v=w}^{\infty}\right)\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}+a_{j+1}+\cdots+a_t=v}}\gamma_k(2\beta)\cdot$$

$$\cdot\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots a_t])\right)^k.$$

Let's investigate the convergence of series

$$\sum_{v=1}^{\infty}\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}+a_{j+1}+\cdots+a_t=v}}\frac{\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots a_t])\right)^k}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}. \tag{12}$$

These series can be ameliorated with the series like

$$K_k\sum_{v=1}^{\infty}v^k\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{j-1}+a_{j+1}+\cdots+a_t=v}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\ldots,a_t\rangle^{2\beta}}=$$

$$=K_k\sum_{v=1}^{\infty}v^k\sum_{s+t=v}\sum_{a_1+\cdots+a_j=s}\frac{1}{\langle a_1,\ldots,a_j\rangle^{2\beta}}\sum_{a_1+\cdots+a_i=t}\frac{1}{\langle a_1,\ldots,a_i\rangle^{2\beta}}.$$

where K_k are some constants. According to lemma 9,

$$\sum_{a_1+\dots+a_j=s} \frac{1}{\langle a_1,\dots,a_j\rangle^{2\beta}} = O\left(\frac{1}{s^{2\beta}}\right),$$

$$\sum_{a_1+\dots+a_i=t} \frac{1}{\langle a_1,\dots,a_i\rangle^{2\beta}} = O\left(\frac{1}{t^{2\beta}}\right),$$

hence, main term of series (12) can be estimated as $O\left(\frac{1}{v^{2\beta-k-1}}\right)$. Thus, series converges when $2\beta-k-1>1$, e. i. $k<2\beta-2$.
When $k<2\beta-2$ constants C'_k can be defined as follows:

$$C'_k = \gamma_k(2\beta)\sum_{v=1}^{\infty} \sum_{a\in A_n, a_1+\dots+a_{j-1}+a_{j+1}+\dots+a_t=v} \frac{(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots a_t]))^k}{\langle a_1,\dots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\dots,a_t\rangle^{2\beta}}.$$

When $k<2\beta-2$, let's estimate diversity

$$C'_k - \gamma_k(2\beta)\sum_{v=1}^{w-1} \sum_{\substack{a\in A_n,\\ a_1+\dots+a_{j-1}+a_{j+1}+\dots+a_t=v}} \frac{(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots a_t]))^k}{\langle a_1,\dots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\dots,a_t\rangle^{2\beta}} =$$

$$= \gamma_k(2\beta)\sum_{v=w}^{\infty} \sum_{\substack{a\in A_n,\\ a_1+\dots+a_{j-1}+a_{j+1}+\dots+a_t=v}} \frac{(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots a_t]))^k}{\langle a_1,\dots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\dots,a_t\rangle^{2\beta}} \ll$$

$$\ll \sum_{v=w}^{\infty} v^k \sum_{s+t=v} \sum_{a_1+\dots+a_j=s} \frac{1}{\langle a_1,\dots,a_j\rangle^{2\beta}} \sum_{a_1+\dots+a_i=t} \frac{1}{\langle a_1,\dots,a_i\rangle^{2\beta}} \ll$$

$$\ll \int_w^{\infty} \frac{dv}{v^{2\beta-k-1}} = O\left(\frac{1}{w^{2\beta-k-2}}\right).$$

Thus we obtain, that kth term when $k<2\beta-2$ is equal to

$$\frac{\gamma_k(2\beta)}{n^{2\beta+k}}\sum_{v=1}^{w-1} \sum_{\substack{a\in A_n,\\ a_1+\dots+a_{j-1}+a_{j+1}+\dots+a_t=v}} \frac{(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots a_t]))^k}{\langle a_1,\dots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\dots,a_t\rangle^{2\beta}} =$$

$$= C'_k\frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta+k}w^{2\beta-k-2}}\right) = C'_k\frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta}w^{2\beta-2}}\right).$$

Now let us consider the error term of the series in case $k\ge 2\beta-2$.
When $k>2\beta-2$

$$R_k = \gamma_k(2\beta)\sum_{v=1}^{w-1} \sum_{\substack{a\in A_n,\\ a_1+\dots+a_{j-1}+a_{j+1}+\dots+a_t=v}} \frac{(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots a_t]))^k}{\langle a_1,\dots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\dots,a_t\rangle^{2\beta}} \le$$

$$\leq \gamma_k(2\beta)\sum_{v=1}^{w-1} v^k \sum_{s+u=v}\ \sum_{a_1+\dots+a_j=s} \frac{1}{\langle a_1,\dots,a_j\rangle^{2\beta}} \sum_{a_1+\dots+a_i=u} \frac{1}{\langle a_1,\dots,a_i\rangle^{2\beta}} \leq$$

$$\leq \gamma_k(2\beta)\cdot 16C_0^2 \int_1^{w-1} \frac{dv}{v^{2\beta-k-1}} \leq$$

$$\leq \gamma_k(2\beta)\cdot 16C_0^2 w^{k+2-2\beta}.$$

Then summing according to $k > 2\beta - 2$, we get .

$$\sum_{k>2\beta-2} \frac{R_k}{n^{2\beta+k}} \leq \sum_{k>2\beta-2} \gamma_k(2\beta)\cdot 16C_0^2 \frac{w^{k+2-2\beta}}{n^{2\beta+k}} =$$

$$= \frac{1}{n^{2\beta}w^{2\beta-2}}\cdot 16\left(C_0\right)^2 \sum_{k>2\beta-2} \gamma_k(2\beta)\cdot\left(\frac{w}{n}\right)^k <$$

$$< \frac{1}{n^{2\beta}w^{2\beta-2}}\cdot 16C_0^2 \sum_{k=1}^{\infty} \gamma_k(2\beta)\cdot\left(\frac{w}{n}\right)^k = \frac{1}{n^{2\beta}w^{2\beta-2}}\cdot 16C_0^2 \left(\frac{1}{1-\frac{w}{n}}\right)^{2\beta}.$$

With the given w value of $\frac{1}{1-\frac{w}{n}}$ doesn't emceed 2, hence the sum can be estimated as $O\left(\frac{1}{n^{2\beta}w^{2\beta-2}}\right)$. 2β can turn out to be integer. In this case when $k = 2\beta - 2$ we get

$$\gamma_k(2\beta)\sum_{v=1}^{w-1} \sum_{\substack{a\in A_n,\\ a_1+\dots+a_{j-1}+a_{j+1}+\dots+a_t=v}} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{2\beta}\langle a_{j+1},\dots,a_t\rangle^{2\beta}} \times$$

$$\times\left(v - \left([a_{i-1},\dots,a_1] + [a_{i+1},\dots a_t]\right)\right)^k = O\left(\log w\right).$$

Thus, the error term in case when 2β is integer is

$$O\left(\frac{\log w}{n^{4\beta-2}} + \frac{1}{w^{2\beta-2}n^{2\beta}}\right),$$

otherwise it is

$$O\left(\frac{1}{w^{2\beta-2}n^{2\beta}}\right).$$

Thus, for $\sum_{n,1}^{(2)}$ the following asymptotic holds: when 2β is integer

$$\Sigma_{n,1}^{(2)} = \frac{C_0}{n^{2\beta}} + O\left(\frac{1}{n^{2\beta}w^{2\beta-2}}\right) +$$

$$+\sum_{1\leq k<2\beta-2}\left({C'}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta}w^{2\beta-2}}\right)\right) +$$

$$+O\left(\frac{\log n}{n^{4\beta-2}} + \frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{1}{n^{2r(\beta-1)}}\right) =$$

$$= \frac{C_0}{n^{2\beta}} + \sum_{1\le k<2\beta-2} C'{}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{\log n}{n^{4\beta-2}} + \frac{1}{n^{2\beta} w^{2(\beta-1)}} + \frac{1}{n^{2r(\beta-1)}}\right)$$

when 2β is not integer ,

$$\Sigma^{(2)}_{n,1} = \frac{C_0}{n^{2\beta}} + O\left(\frac{1}{n^{2\beta} w^{2\beta-2}}\right) +$$

$$+ \sum_{1\le k<2\beta-2} \left(C'{}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta} w^{2\beta-2}}\right)\right) + O\left(\frac{1}{n^{2\beta} w^{2(\beta-1)}} + \frac{1}{n^{2r(\beta-1)}}\right) =$$

$$= \frac{C_0}{n^{2\beta}} + \sum_{1\le k<2\beta-2} C'{}_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta} w^{2(\beta-1)}} + \frac{1}{n^{2r(\beta-1)}}\right).$$

Lemma is proved.

Theorem is proved.

§4. Proof of the main result.

Let's remind, that r and w are integer parameters, satisfying conditions $r \ge 1$ and $1 \le w \le n$.
We'll use the partition of A_n, defined in the proof of theorem 3. Then, we divide $A^{(2)}_{n,1}$ into 2 sets: $A^{(3)}_{n,1}$, where the greatest partial quotient is the last one, and $A^{(3)}_{n,2}$, where it's not the last one:

$$A^{(3)}_{n,1} = \{a \in A^{(2)}_{n,1} \mid a_t > \max\{a_1, \dots, a_{t-1}\}\}$$

and

$$A^{(3)}_{n,2} = A^{(2)}_{n,1} \backslash A^{(3)}_{n,1} = \{a = (a_1, \dots, a_t) \in A^{(2)}_{n,1} \mid a_t \le \max\{a_1, \dots, a_{t-1}\}\}.$$

For subset $A^{(j)}_{n,i}$ of A let's define

$$\Sigma^{(j)}_{n,i} = \sum_{a\in A^{(j)}_{n,i}} \frac{1}{(qq_-)^{\beta}} + \frac{1}{(qq_+)^{\beta}},$$

where $a = (a_1, \dots, a_t), q = q(a) = \langle a_1, \dots, a_t\rangle$; $q_- = q_-(a)$ and $q_+ = q_+(a)$ are defined in lemma 1.
Then we divide $\sum^{(3)}_{n,1}$ into $\sum^{(3)+}_{n,1}$ and $\sum^{(3)-}_{n,1}$ with

$$\Sigma^{(3)+}_{n,1} = \sum_{a\in A^{(3)}_{n,1}} \frac{1}{(qq_+)^{\beta}}, \Sigma^{(3)-}_{n,1} = \sum_{a\in A^{(3)}_{n,1}} \frac{1}{(qq_-)^{\beta}}.$$

Thus,

$$\sigma(F_n) = \Sigma^{(1)}_{n,2} + \Sigma^{(2)}_{n,2} + \Sigma^{(3)}_{n,2} + \Sigma^{(3)+}_{n,1} + \Sigma^{(3)-}_{n,1}. \tag{13}$$

Let's estimate these sums separately.
According to [5], for $\sum_{n,2}^{(1)}$ the following estimate holds:
Lemma 11.
$$\Sigma_{n,2}^{(1)} \leq \frac{1}{n^{(\beta-1)(2r-1)}}. \tag{14}$$

Lemma 12.
When $n \to \infty$ we have
$$\Sigma_{n,2}^{(2)} \ll \frac{n^2 \log^{3\beta} n}{w^{3\beta}}. \tag{15}$$
Lemma 12 is an analogue of lemma 5.
Proof.
According to lemma 4 for every $a \in A_{n,1}^{(1)}$ holds $t \leq Kr \log n$. As $n = a_1 + \ldots + a_t \leq t \max a_j$, then $\max a_j \geq \frac{n}{Kr \log n}$.
Let $a \in A_{n,2}^{(2)}$ and j be such, that $a_j = \max\{a_1, \ldots, a_t\}$. As $a_j \leq n - w$, then for the sum of other a_j we have $\sum\limits_{i \neq j} a_j \geq w$, and, similarly to the above, we have $\max_{i \neq j} a_j \geq \frac{w}{Kr \log n}$.
Thus, there's at least one index $j \leq t-1$ such, that $a_j \geq \frac{w}{Kr \log n}$. Hence,

$$\Sigma_{n,2}^{(2)} \leq \sum_{\substack{a_1+\ldots+a_t=n,\\ \langle a_1,\ldots,a_t\rangle \leq n^r,\\ \exists k,l,k\neq l:\, a_k,a_l \geq \frac{w}{K\log n}}} \left(\frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right) \leq$$

$$\leq \sum_{\substack{a_1+\ldots+a_t=n,\\ \langle a_1,\ldots,a_t\rangle \leq n^r,\\ \exists k\leq (t-1):\, a_k,a_t \geq \frac{w}{K\log n}}} \left(\frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right) +$$

$$+ \sum_{\substack{a_1+\ldots+a_t=n,\\ \langle a_1,\ldots,a_t\rangle \leq n^r,\\ \exists k,l\leq (t-1):\, a_k,a_l \geq \frac{w}{K\log n}}} \left(\frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right) \leq$$

$$\leq 2 \sum_{\substack{a_1+\ldots+a_t=n,\\ \langle a_1,\ldots,a_t\rangle \leq n^r,\\ \exists k\leq (t-1):\, a_k,a_t \geq \frac{w}{K\log n}}} \frac{1}{(qq_-)^\beta} +$$

$$+2 \sum_{\substack{a_1+\ldots+a_t=n,\\ \langle a_1,\ldots,a_t\rangle \le n^r,\\ \exists k,l\le (t-1):\, a_k,a_l \ge \frac{w}{K\log n}}} \frac{1}{(qq_-)^\beta}.$$

Using (4), we obtain the following estimation for the continuant:

$$q(a) \ge a_k a_t \langle a_1,..,a_{k-1}\rangle\langle a_{k+1},..,a_{t-1}\rangle$$

and

$$q(a) \ge a_k a_l \langle a_1,..,a_{k-1}\rangle\langle a_{k+1},..,a_{l-1}\rangle\langle a_{l+1},..,a_t\rangle$$

Thus, splitting sum $\sum_{n,2}^{(} 2)$ into two parts (one part corresponds to items with big last patial quotient a_t, and another part corresponds to items which have big partial quotints a_k, a_l,such that neither of them is the last one), we have

$$\Sigma_{n,2}^{(2)} \ll \sum_{\substack{a_k+a_t\le n,\\ a_k,a_t\ge \frac{w}{K\log n}}} \frac{1}{a_k^{2\beta}a_t^{\beta}} \sum_{u+v=n-a_k-a_t}$$

$$\sum_{a_1+\cdots+a_{k-1}=u} \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{2\beta}} \sum_{a_{k+1}+\cdots+a_{t-1}=v} \frac{1}{\langle a_{k+1},\ldots,a_{t-1}\rangle^{2\beta}}$$

$$+ \sum_{\substack{a_k+a_l\le n,\\ a_k,a_l\ge \frac{w}{K\log n}}} \frac{1}{a_k^{2\beta}a_l^{2\beta}} \sum_{u+v+s=n-a_k-a_l} \sum_{a_1+\cdots+a_{k-1}=u} \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{2\beta}}$$

$$\sum_{a_{k+1}+\cdots+a_{l-1}=v} \frac{1}{\langle a_{k+1},\ldots,a_{l-1}\rangle^{2\beta}} \sum_{a_{k+1}+\cdots+a_{l-1}=s} \frac{1}{\langle a_{l+1},\ldots,a_{t-1}\rangle^{\beta}\langle a_{l+1},\ldots,a_t\rangle^{\beta}}. \tag{16}$$

Let's consider the inner sum in the first item in (16) .

$$\sum_{u+v=n-a_k-a_t} \sum_{a_1+\cdots+a_{k-1}=u} \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{2\beta}} \sum_{a_{k+1}+\cdots+a_{t-1}=v} \frac{1}{\langle a_{k+1},\ldots,a_{t-1}\rangle^{2\beta}} =$$

$$\sum_{u+v=n-a_k-a_t} \sum_{x_1+\cdots+x_r=u} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \le$$

$$\le \sum_{u+v\le n} \sum_{x_1+\cdots+x_r=u} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \le$$

$$\le \sum_{u+v\le \infty} \sum_{x_1+\cdots+x_r=u} \frac{1}{\langle x_1,\ldots,x_r\rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1,\ldots,y_h\rangle^{2\beta}} \le$$

$$\le \left(1 + 2 \sum_{\substack{x_1 + \cdots + x_r \le \infty, \\ x_r \ge 2}} \frac{1}{\langle x_1, \ldots, x_r \rangle^{2\beta}} \right)^2 \le$$

$$\le \left(1 + 2 \frac{\zeta(2\beta - 1)}{\zeta 2\beta} \right)^2.$$

Then for the outer sum for the fist item in (16) holds

$$\sum_{\substack{a_k + a_t \le n, \\ a_k, a_t \ge \frac{w}{K \log n}}} \frac{1}{(a_k)^{2\beta} (a_t)^{\beta}} \ll \frac{\log^{3\beta} n}{w^{3\beta}} \left(n - 2 \frac{w}{K \log n} \right)^2.$$

Here $\left(n - 2 \frac{w}{K \log n} \right)^2$ is the number of elements in sum, $\frac{\log^{3\beta} n}{w^{3\beta}}$ – upper bound for the value under summation. Now let's consider the inner sum in the second item in (16).

$$\sum_{u+v+s=n-a_k-a_l} \sum_{a_1+\cdots+a_{k-1}=u} \frac{1}{\langle a_1, \ldots, a_{k-1} \rangle^{2\beta}} \times$$

$$\times \sum_{a_{k+1}+\cdots+a_{t-l}=v} \frac{1}{\langle a_{k+1}, \ldots, a_{t-1} \rangle^{2\beta}} \times$$

$$\times \sum_{a_{k+1}+\cdots+a_{l-1}=s} \frac{1}{\langle a_{l+1}, \ldots, a_{t-1} \rangle^{\beta} \langle a_{l+1}, \ldots, a_t \rangle^{\beta}} =$$

$$= \sum_{u+v+s=n-a_k-a_l} \sum_{x_1+\cdots+x_j=u} \frac{1}{\langle x_1, \ldots, x_j \rangle^{2\beta}} \times$$

$$\times \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1, \ldots, y_h \rangle^{2\beta}} \sum_{z_1+\cdots+z_p=s} \frac{1}{\langle z_1, \ldots, z_{p-1} \rangle^{\beta} \langle z_1, \ldots, z_p \rangle^{\beta}} \le$$

$$\le \sum_{u+v+s \le n} \sum_{x_1+\cdots+x_j=u} \frac{1}{\langle x_1, \ldots, x_j \rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1, \ldots, y_h \rangle^{2\beta}} \times$$

$$\times \sum_{z_1+\cdots+z_p=s} \frac{1}{\langle z_1, \ldots, z_{p-1} \rangle^{\beta} \langle z_1, \ldots, z_p \rangle^{\beta}} \le$$

$$\le \sum_{u+v+s \le \infty} \sum_{x_1+\cdots+x_j=u} \frac{1}{\langle x_1, \ldots, x_j \rangle^{2\beta}} \sum_{y_1+\cdots+y_h=v} \frac{1}{\langle y_1, \ldots, y_h \rangle^{2\beta}} \times$$

$$\times \sum_{z_1+\cdots+z_p=s} \frac{1}{\langle z_1, \ldots, z_{p-1} \rangle^{\beta} \langle z_1, \ldots, z_p \rangle^{\beta}} \le$$

$$\leq \left(1+2\sum_{\substack{x_1+\dots+x_j\leq\infty,\\ x_j\geq 2}}\frac{1}{\langle x_1,\dots,x_j\rangle^{2\beta}}\right)^2\times$$

$$\times\left(1+\sum_{\substack{z_1+\dots+z_p\leq\infty,\\ z_p\geq 2}}\frac{1}{\langle z_1,\dots,z_{p-1}\rangle^{\beta}\langle z_1,\dots,z_p\rangle^{\beta}}\right)\leq$$

$$\leq\left(1+2\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right)^2\left(1+\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right).$$

Then for the outer sum in the second item in (16) the following estimation holds

$$\sum_{\substack{a_k+a_t\leq n,\\ a_k,a_l\geq\frac{w}{K\log n}}}\frac{1}{(a_k)^{2\beta}(a_l)^{2\beta}}\ll\frac{\log^{4\beta}n}{w^{4\beta}}\left(n-2\frac{w}{K\log n}\right)^2.$$

Here $\left(n-2\frac{w}{K\log n}\right)^2$ is the number of elements in sum, $\frac{\log^{4\beta}n}{w^{4\beta}}$ -upper bound for the value under summation.
Thus, for $\sum_{n,2}^{(2)}$ we get

$$\Sigma_{n,2}^{(2)}=O\left(\frac{n^2\log^{3\beta}n}{w^{3\beta}}\right).$$

Lemma is proved.

Lemma 13.
When $w\leq\frac{n}{2}-2$ the following asymptotic holds for the sum $\sum_{n,2}^{(3)}$:
in the case β is integer,

$$\Sigma_{n,2}^{(3)}=\sum_{0\leq k<\beta-2}B_k\frac{1}{n^{2\beta+k}}+O\left(\frac{1}{n^{(\beta-1)(2r-1)}}+\frac{1}{n^{2\beta}w^{\beta-2}}+\frac{\log w}{n^{3\beta-2}}\right),\qquad(17)$$

otherwise,

$$\Sigma_{n,2}^{(3)}=\sum_{0\leq k<\beta-2}B_k\frac{1}{n^{2\beta+k}}+O\left(\frac{1}{n^{(\beta-1)(2r-1)}}+\frac{1}{n^{2\beta}w^{\beta-2}}\right),\qquad(18)$$

where B_k are some constants.

Proof.

$$\Sigma_{n,2}^{(3)} = \sum_{\substack{a\in A_n,\\ q(a)<n^r,\\ a_i>n-w, j\neq t}} \left(\frac{1}{(qq_-)^\beta}+\frac{1}{(qq_+)^\beta}\right) =$$

$$= \sum_{\substack{a\in A_n,\\ a_i>n-w, j\neq t}} \left(\frac{1}{(qq_-)^\beta}+\frac{1}{(qq_+)^\beta}\right) - \sum_{\substack{a\in A_n,\\ q(a)\geq n^r,\\ a_i>n-w, j\neq t}} \left(\frac{1}{(qq_-)^\beta}+\frac{1}{(qq_+)^\beta}\right).$$

The second sum can be estimated according to lemma 11 as

$$\sum_{\substack{a\in A_n,\\ q(a)\geq n^r,\\ a_i>n-w, j\neq t}} \left(\frac{1}{(qq_-)^\beta}+\frac{1}{(qq_+)^\beta}\right) = O\left(\frac{1}{n^{(\beta-1)(2r-1)}}\right).$$

Let us consider the first sum. Let $a_i = n-v, v = 1,\ldots,(w-1)$. Using (4) we get

$$\langle a_1,\ldots,a_i,\ldots,a_t\rangle = \langle a_1,\ldots,a_{j-1}\rangle\langle a_{j+1},\ldots,a_t\rangle\left(a_i+[a_{j-1},\ldots,a_1]+[a_{j+1},\ldots,a_t]\right) =$$

$$= n\langle a_1,\ldots,a_{j-1}\rangle\langle a_{j+1},\ldots,a_t\rangle\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{j-1},\ldots,a_1]+[a_{j+1},\ldots,a_t]\right)\right),$$

$$\langle a_1,\ldots,a_i,\ldots,a_{t-1}\rangle = n\langle a_1,\ldots,a_{i-1}\rangle\langle a_{i+1},\ldots,a_{t-1}\rangle\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_{t-1}]\right)\right),$$

$$\langle a_1,\ldots,a_i,\ldots,a_t-1\rangle = n\langle a_1,\ldots,a_{i-1}\rangle\langle a_{i+1},\ldots,a_t-1\rangle\cdot$$
$$\cdot\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t-1]\right)\right).$$

Then for $\frac{1}{(qq_-)^\beta}$ and $\frac{1}{(qq_+)^\beta}$ we can obtain the following formulas:

$$\frac{1}{(qq_-)^\beta} = \frac{1}{n^{2\beta}}\frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}}\frac{1}{\left(\langle a_{i+1},\ldots,a_t\rangle\langle a_{i+1},\ldots,a_{t-1}\rangle\right)^\beta}\cdot$$
$$\cdot\frac{1}{\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t]\right)\right)^\beta\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_{t-1}]\right)\right)^\beta}, \tag{19}$$

$$\frac{1}{(qq_+)^\beta} = \frac{1}{n^{2\beta}}\frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}}\frac{1}{\left(\langle a_{i+1},\ldots,a_t\rangle\langle a_{i+1},\ldots,a_t-1\rangle\right)^\beta}\cdot$$
$$\cdot\frac{1}{\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t]\right)\right)^\beta\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t-1]\right)\right)^\beta}. \tag{20}$$

Let's expand

$$\frac{1}{\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t]\right)\right)^{\beta}}$$

and

$$\frac{1}{\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}]\right)\right)^{\beta}}$$

into Taylor series according to parameters

$$\frac{v}{n}-\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t]\right)$$

and

$$\frac{v}{n}-\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}]\right)$$

correspondingly. Thus we obtain

$$\begin{aligned}\frac{1}{\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t]\right)\right)^{\beta}}&=\\ =1+\sum_{k=1}^{\infty}\frac{1}{n^k}\gamma_k(\beta)\cdot\left(v-\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t]\right)\right)^k&\end{aligned}\tag{21}$$

when $|\frac{v}{n}-\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t]\right)|<1$,

$$\begin{aligned}\frac{1}{\left(1-\frac{v}{n}+\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}]\right)\right)^{\beta}}&=\\ =1+\sum_{k=1}^{\infty}\frac{1}{n^k}\gamma_k(\beta)\cdot\left(v-\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}]\right)\right)^k.&\end{aligned}\tag{22}$$

when $|\frac{v}{n}-\frac{1}{n}\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}]\right)|<1$, where $\gamma_k(\beta)$ are defined in (10). When $v\le n-1$ series converge absolutely.

Substituting (21) and (22) into (19), we get with the given v

$$\begin{aligned}\frac{1}{(qq_-)^{\beta}}=\frac{1}{n^{2\beta}\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\cdot\\ \cdot\,(1+\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{l+m=k}\gamma_l(\beta)\left(v-\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}]\right)\right)^l\cdot\\ \cdot\gamma_m(\beta)\left(v-\left([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t]\right)\right)^m\ ).\end{aligned}\tag{23}$$

Substituting the obtained result for $\frac{1}{(qq_-)^{\beta}}$ into

$$\sum_{v=1}^{w-1}\ \sum_{a\in A_n,a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v}\frac{1}{(qq_-)^{\beta}}=$$

$$= \sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{n^{2\beta}\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\cdot$$

$$\cdot\Big(1+\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{l+m=k}\gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}])\right)^l\cdot$$

$$\cdot\gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m\Big)=$$

$$= \frac{1}{n^{2\beta}}\sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{n^{2\beta+k}}\sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\cdot$$

$$\cdot\sum_{l+m=k}\gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}])\right)^l\cdot\gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m.$$

Let's investigate sum at $\frac{1}{n^{2\beta+k}}$.

$$R_k^- = \sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\cdot$$

$$\cdot\sum_{l+m=k}\gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}])\right)^l\cdot\gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m=$$

$$=\left(\sum_{v=1}^{\infty}-\sum_{v=w}^{\infty}\right) \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\cdot$$

$$\cdot\sum_{l+m=k}\gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}])\right)^l\cdot\gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m.$$

Investigate convergence of the first series.

$$\sum_{v=1}^{\infty} \sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v}} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}$$

$$\sum_{l+m=k}\gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}])\right)^l\cdot\gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m\ll$$

$$\ll\sum_{v=1}^{\infty}v^k \sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v}} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\ll$$

$$\ll\sum_{v=1}^{\infty}v^k\sum_{\nu+\eta=v}\sum_{a_1+\cdots+a_{i-1}=\nu}\frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}}\sum_{a_{i+1}+\cdots+a_t=\eta}\frac{1}{\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}\ll$$

$$\ll \sum_{v=1}^{\infty} v^k \sum_{\nu+\eta=v} \sum_{a_1+\cdots+a_t=\nu} \frac{1}{q(a)^{2\beta}} \sum_{a_1+\cdots+a'_t=\eta} \frac{1}{(q(a)\,q_-(a))^{\beta}}.$$

It follows from lemma 9, that

$$\sum_{a_1+\cdots+a_t=\nu} \frac{1}{q(a)^{2\beta}} = O\left(\frac{1}{\nu^{2\beta}}\right),$$

it follows from lemma in [3] that

$$\sum_{a_1+\cdots+a'_t=\eta} \frac{1}{(q(a)\,q_-(a))^{\beta}} = O\left(\frac{1}{\eta^{\beta}}\right).$$

Thus, common term of the series can be estimated as $O\left(\frac{1}{v^{\beta-k-1}}\right)$, so the series converges when $\beta-k-1>1$, i.e. when $k<\beta-2$.
With this k let us estimate the error term of the series:

$$\sum_{v=w}^{\infty} \sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v}} \frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\ldots,a_t\rangle^{\beta}\langle a_{i+1},\ldots,a_{t-1}\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_{t-1}])\right)^l \cdot$$

$$\cdot\gamma_m(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t])\right)^m \ll$$

$$\ll \sum_{v=w}^{\infty} \sum_{\nu+\eta=v} \sum_{a_1+\cdots+a_t=\nu} \frac{1}{q(a)^{2\beta}} \sum_{a_1+\cdots+a'_t=\eta} \frac{1}{(q(a)\,q_-(a))^{\beta}} \ll$$

$$\ll \int_w^{\infty} \frac{dv}{v^{\beta-k-1}} = O\left(\frac{1}{w^{\beta-k-2}}\right).$$

Thus, coefficient at kth term for $k<\beta-2$ is equal to

$$R_k^- = B_k^- + O\left(\frac{1}{w^{\beta-k-2}}\right),$$

where

$$B_k^- = \sum_{v=1}^{\infty} \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\ldots,a_t\rangle^{\beta}\langle a_{i+1},\ldots,a_{t-1}\rangle^{\beta}}\cdot$$

$$\cdot\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_{t-1}])\right)^l \cdot \gamma_m(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t])\right)^m.$$

Now let's consider the terms when $k \geq \beta-2$. Let's get the estimation for R_k^-.

$$R_k^- \leq \sum_{v=1}^{w-1} v^k \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta)$$

$$\sum_{\substack{a \in A_n, \\ a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v}} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}} \leq$$

$$\leq \sum_{v=1}^{w-1} v^k \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) \sum_{s+u=v} 8C_0 \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \frac{1}{s^{2\beta}u^{\beta}} \leq$$

$$\leq 8C_0 \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) w^{k-\beta+2}.$$

Then for residual series we get the following estimation:

$$\sum_{k>\beta-2} \frac{R_k^-}{n^{2\beta+k}} \leq \sum_{k>\beta-2} \frac{1}{n^{2\beta+k}} 8C_0 \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) w^{k-\beta+2} =$$

$$= 8C_0 \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \frac{1}{n^{2\beta}w^{\beta-2}} \sum_{k>\beta-2} \frac{w^k}{n^k} \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) \leq$$

$$\leq 8C_0 \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \frac{1}{n^{2\beta}w^{\beta-2}} \sum_{k=1}^{\infty} \frac{w^k}{n^k} \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) =$$

$$= 8C_0 \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \frac{1}{n^{2\beta}w^{\beta-2}} \left(\frac{1}{1-\frac{w}{n}}\right)^{2\beta}.$$

With the given w magnitude $\frac{1}{1-\frac{w}{n}}$ doesn't exceed 2, thus , residual series can be estimated as $O(\frac{1}{n^{2\beta}w^{\beta-2}})$. If β is integer, then for $k=\beta-2$ we obtain

$$\sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_{t-1}\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_{t-1}])\right)^l \cdot$$

$$\cdot \gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m = O(\log w).$$

Similar actions can be made for part of the sum with $\frac{1}{qq_+}$.
We get

$$\sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v} \frac{1}{(qq_+)^{\beta}} =$$

$$= \frac{1}{n^{2\beta}} \sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_t-1\rangle^{\beta}} +$$

$$\sum_{k=1}^{\infty} \frac{1}{n^{2\beta+k}} \sum_{v=1}^{w-1} \sum_{a\in A_n, a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_t-1\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t-1])\right)^l \cdot \gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m .$$

Coefficient R_k^+ at kth term equals

$$R_k^+ = (\sum_{v=1}^{\infty} - \sum_{v=w}^{\infty}) \sum_{a\in A_n, a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_t-1\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t-1])\right)^l \cdot \gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m .$$

Let us consider the first series. It can be majorized with th following series:

$$\sum_{v=1}^{\infty} v^k \sum_{\nu+\eta=v} \sum_{a_1+\dots+a_t=\nu} \frac{1}{q(a)^{2\beta}} \sum_{a_1+\dots+a'_t=\eta} \frac{1}{(q(a)\,q_+(a))^{\beta}}.$$

According to lemma 9,

$$\sum_{a_1+\dots+a'_t=\nu} \frac{1}{(q(a))^{2\beta}} = O\left(\frac{1}{\nu^{2\beta}}\right),$$

and it follows from lemma 6 in [3] that

$$\sum_{a_1+\dots+a'_t=\eta} \frac{1}{(q(a)\,q_+(a))^{\beta}} = O\left(\frac{1}{\eta^{2\beta}}\right),$$

then common term of this series can be estimated as $O\left(\frac{1}{v^{2\beta-k-1}}\right)$, so, the series converges when $k<2\beta-2$.
With this k let us estimate the residual series:

$$\sum_{v=w}^{\infty} \sum_{a\in A_n, a_1+\dots+a_{i-1}+a_{i+1}+\dots+a_t=v} \frac{1}{\langle a_1,\dots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\dots,a_t\rangle^{\beta}\langle a_{i+1},\dots,a_t-1\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t-1])\right)^l \cdot$$

$$\cdot \gamma_m(\beta)\left(v-([a_{i-1},\dots,a_1]+[a_{i+1},\dots,a_t])\right)^m \ll$$

$$\ll \sum_{v=w}^{\infty} \sum_{\nu+\eta=v} \sum_{a_1+\dots+a_t=\nu} \frac{1}{q(a)^{2\beta}} \sum_{a_1+\dots+a'_t=\eta} \frac{1}{(q(a)\,q_+(a))^{\beta}} \ll$$

$$\ll \int_w^{\infty} \frac{dv}{v^{2\beta-k-1}} = O\left(\frac{1}{w^{2\beta-k-2}}\right).$$

Thus, coefficient R_k^+ at kth term when $k<2\beta-2$ equals

$$R_k^+ = B_k^+ + O\left(\frac{1}{w^{2\beta-k-2}}\right),$$

where

$$B_k^+ = \sum_{v=1}^{\infty} \sum_{a\in A_n, a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v} \frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\ldots,a_t\rangle^{\beta}\langle a_{i+1},\ldots,a_t-1\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t-1])\right)^l \cdot \gamma_m(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t])\right)^m .$$

Now let's consider the terms when $k \geq 2\beta-2$. Let's get the estimation for kth term .

$$R_k^+ \leq \sum_{v=1}^{w-1} v^k \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta)$$

$$\sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v}} \frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\ldots,a_t\rangle^{\beta}\langle a_{i+1},\ldots,a_t-1\rangle^{\beta}} \leq$$

$$\leq \sum_{v=1}^{w-1} v^k \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) \sum_{s+u=v} 4C_0 \frac{1}{s^{2\beta}u^{2\beta}} \leq$$

$$\leq 4C_0 \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) w^{k-2\beta+2}.$$

Then we get the following estimation for the residual series.

$$\sum_{k>2\beta-2} \frac{R_k^+}{n^{2\beta+k}} \leq \sum_{k>2\beta-2} \frac{1}{n^{2\beta+k}} 4\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right)^2\right) \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) w^{k-2\beta+2} =$$

$$= 4C_0 \frac{1}{n^{2\beta}w^{2\beta-2}} \sum_{k>2\beta-2} \frac{w^k}{n^k} \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) \leq$$

$$\leq 4C_0 \frac{1}{n^{2\beta}w^{2\beta-2}} \sum_{k=1}^{\infty} \frac{w^k}{n^k} \sum_{l+m=k} \gamma_l(\beta)\gamma_m(\beta) \leq$$

$$\leq 4C_0 \frac{1}{n^{2\beta}w^{\beta-2}} \left(\frac{1}{1-\frac{w}{n}}\right)^{2\beta}.$$

With the given w magnitude $\frac{1}{1-\frac{w}{n}}$ doesn't exceed 2, so, residual series can be estimated as $O(\frac{1}{n^{2\beta}w^{2\beta-2}})$. If 2β is integer, then for $k=2\beta-2$ we get

$$\sum_{v=1}^{w-1} \sum_{\substack{a\in A_n,\\ a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_t=v}} \frac{1}{\langle a_1,\ldots,a_{i-1}\rangle^{2\beta}\langle a_{i+1},\ldots,a_t\rangle^{\beta}\langle a_{i+1},\ldots,a_t-1\rangle^{\beta}}$$

$$\sum_{l+m=k} \gamma_l(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t-1])\right)^l \cdot$$

$$\cdot\gamma_m(\beta)\left(v-([a_{i-1},\ldots,a_1]+[a_{i+1},\ldots,a_t])\right)^m = O(\log w).$$

Thus, adding sum for $\frac{1}{q(a)q_-(a)}$ to sum for $\frac{1}{q(a)q_+(a)}$, we obtain when β is integer

$$\Sigma_{n,2}^{(3)} = \sum_{0\le k<\beta-2} B_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{(\beta-1)(2r-1)}} + \frac{\log w}{n^{3\beta-2}} + \frac{1}{w^{\beta-2}n^{2\beta}}\right),$$

when β is not integer

$$\Sigma_{n,2}^{(3)} = \sum_{0\le k<\beta-2} B_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{(\beta-1)(2r-1)}} + \frac{1}{w^{\beta-2}n^{2\beta}}\right),$$

where

$$B_k = B_k^- + B_k^+.$$

Lemma is proved.

Lemma 14. *When 2β is integer*

$$\Sigma_{n,1}^{(3)+} = \frac{1}{n^{2\beta}}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-1} D_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{\log w}{n^{4\beta-1}} + \frac{1}{n^{(\beta-1)(2r-1)}} + \frac{1}{w^{2\beta-1}n^{2\beta}}\right), \tag{24}$$

when 2β is not integer

$$\Sigma_{n,1}^{(3)+} = \frac{1}{n^{2\beta}}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-1} D_k \frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{1}{n^{(\beta-1)(2r-1)}} + \frac{1}{w^{2\beta-1}n^{2\beta}}\right). \tag{25}$$

where D_k are some constants.

Proof.

$$\Sigma_{n,1}^{(3)+} = \sum_{a\in A_n, q(a)<n^r, a_t>n-w} \frac{1}{(qq_+)^\beta}$$

$$= \sum_{a\in A_n, a_t>n-w} \frac{1}{(qq_+)^\beta} - \sum_{a\in A_n, q(a)\ge n^r} \frac{1}{(qq_+)^\beta}$$

The second sum can be estimated according to lemma 11 as

$$\sum_{a\in A_n, q(a)\ge n^r} \frac{1}{(qq_+)^\beta} = O\left(\frac{1}{n^{(\beta-1)(2r-1)}}\right).$$

For the first sum we have

$$\sum_{a\in A_n, a_t>n-w} \frac{1}{(qq_+)^\beta} = \sum_{v=1}^{w} \sum_{a\in A_n, a_1+\ldots+a_{t-1}=v} \frac{1}{(qq_+)^\beta}. \tag{26}$$

Here $a_t = n-v, v=1,..,w$. As

$$q = a_t q_- + (q_-)_- = q_-(n-v+[a_{t-1},\ldots,a_1]),$$

$$q_+ = (a_t - 1)\, q_- + (q_-)_- = q_- \left(n - v - 1 + [a_{t-1}, \dots, a_1]\right),$$

then we get for $\frac{1}{(qq_+)^\beta}$

$$\frac{1}{(qq_+)^\beta} = \frac{1}{n^{2\beta} q_-^{2\beta}} \left(\frac{1}{\left(1 - \frac{v}{n} + \frac{1}{n}[a_{t-1}, \dots, a_1]\right)^\beta} \cdot \frac{1}{\left(1 - \frac{v}{n} - \frac{1}{n} + \frac{1}{n}[a_{t-1}, \dots, a_1]\right)^\beta} \right). \tag{27}$$

Expanding magnitudes $\frac{1}{\left(1 - \frac{v}{n} + \frac{1}{n}[a_{t-1}, \dots, a_1]\right)^\beta}$ and $\frac{1}{\left(1 - \frac{v}{n} - \frac{1}{n} + \frac{1}{n}[a_{t-1}, \dots, a_1]\right)^\beta}$ into Teilor series according to parameters $\frac{v}{n} - \frac{1}{n}[a_{t-1}, \dots, a_1]$ we obtain $\frac{v}{n} + \frac{1}{n} - \frac{1}{n}[a_{t-1}, \dots, a_1]$ correspondingly , when $v \le n - 1$

$$\frac{1}{\left(1 - \frac{v}{n} + \frac{1}{n}[a_{t-1}, \dots, a_1]\right)^\beta} = 1 + \sum_{k=1}^{\infty} \gamma_k(\beta) \left(\frac{v}{n} - \frac{1}{n}[a_{t-1}, \dots, a_1] \right)^k,$$

$$\frac{1}{\left(1 - \frac{v}{n} - \frac{1}{n} + \frac{1}{n}[a_{t-1}, \dots, a_1]\right)^\beta} = 1 + \sum_{k=1}^{\infty} \gamma_k(\beta) \left(\frac{v}{n} + \frac{1}{n} - \frac{1}{n}[a_{t-1}, \dots, a_1] \right)^k,$$

where $\gamma_k(\beta)$ are defined in (10). Then, substituting obtained series into (27), we get when $v \le n - 1$

$$\frac{1}{(qq_+)^\beta} =$$

$$= \frac{1}{n^{2\beta} q_-^{2\beta}} \left(1 + \sum_{k=1}^{\infty} \sum_{l+m=k} \gamma_l(\beta) \left(\frac{v}{n} - \frac{1}{n}[a_{t-1}, \dots, a_1] \right)^l \cdot \gamma_m(\beta) \left(\frac{v}{n} + \frac{1}{n} - \frac{1}{n}[a_{t-1}, \dots, a_1] \right)^m \right) =$$

$$= \frac{1}{n^{2\beta} q_-^{2\beta}} \left(1 + \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{l+m=k} \gamma_l(\beta) \left(v - [a_{t-1}, \dots, a_1]\right)^l \cdot \gamma_m(\beta) \left(v + 1 - [a_{t-1}, \dots, a_1]\right)^m \right).$$

Next, substituting expression for $\frac{1}{(qq_+)^\beta}$ into (26), we obtain

$$\sum_{v=1}^{w-1} \sum_{a_1 + \dots + a_{t-1} = v} \frac{1}{(qq_+)^\beta} = \sum_{v=1}^{w-1} \sum_{a_1 + \dots + a_{t-1} = v} \frac{1}{n^{2\beta} q_-^{2\beta}} \cdot$$

$$\cdot \left(1 + \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{l+m=k} \gamma_l(\beta) \left(v - [a_{t-1}, \dots, a_1]\right)^l \cdot \gamma_m(\beta) \left(v + 1 - [a_{t-1}, \dots, a_1]\right)^m \right) =$$

$$= \frac{2}{n^{2\beta}} \sum_{v=1}^{w-1} \sum_{a_1 + \dots + a_{t-1} = v, a_{t-1} \ge 2} \frac{1}{q_-^{2\beta}} + \sum_{k=1}^{\infty} \frac{2}{n^{2\beta+k}} \sum_{v=1}^{w-1} \sum_{a_1 + \dots + a_{t-1} = v, a_{t-1} \ge 2} \frac{1}{q_-^{2\beta}} \cdot$$

$$\cdot \sum_{l+m=k} \gamma_l(\beta) \left(v - [a_{t-1}, \dots, a_1]\right)^l \cdot \gamma_m(\beta) \left(v + 1 - [a_{t-1}, \dots, a_1]\right)^m.$$

Let's consider sum R_k at $\frac{1}{n^{2\beta+k}}$.

$$R_k = \sum_{v=1}^{w-1} \sum_{a_1 + \dots + a_{t-1} = v, a_{t-1} \ge 2} \frac{2}{q_-^{2\beta}} \sum_{l+m=k} \gamma_l(\beta) \left(v - [a_{t-1}, \dots, a_1]\right)^l \cdot \gamma_m(\beta) \left(v + 1 - [a_{t-1}, \dots, a_1]\right)^m =$$

$$= \left(\sum_{v=1}^{\infty} - \sum_{v=w}^{\infty}\right) \sum_{a_1+\cdots+a_{t-1}=v, a_{t-1}\geq 2} \frac{2}{q_-^{2\beta}} \sum_{l+m=k} \gamma_l(\beta)\left(v - [a_{t-1},\ldots,a_1]\right)^l \cdot \gamma_m(\beta)\left(v+1-[a_{t-1},\ldots,a_1]\right)^m .$$

Let us consider the first sum:

$$\sum_{v=1}^{\infty} \sum_{a_1+\cdots+a_{t-1}=v, a_{t-1}\geq 2} \frac{2}{q_-^{2\beta}} \sum_{l+m=k} \gamma_l(\beta)\left(v - [a_{t-1},\ldots,a_1]\right)^l \cdot \gamma_m(\beta)\left(v+1-[a_{t-1},\ldots,a_1]\right)^m \ll$$

$$\ll \sum_{v=1}^{\infty} v^k \sum_{a_1+\cdots+a_{t-1}=v, a_{t-1}\geq 2} \frac{1}{q_-^{2\beta}}.$$

According to lemma 9

$$\sum_{a_1+\cdots+a_t=\nu} \frac{1}{q\,(a)^{2\beta}} = O\left(\frac{1}{\nu^{2\beta}}\right),$$

thus, common term of the given series is $O\left(\frac{1}{n^{2\beta-k}}\right)$, so series converges when $2\beta - k > 1$, the given when $k < 2\beta - 1$. Let us estimate residual series with these k :

$$\sum_{v=w}^{\infty} \sum_{a_1+\cdots+a_{t-1}=v} \frac{2}{q_-^{2\beta}} \sum_{l+m=k} \gamma_l(\beta)\left(v - [a_{t-1},\ldots,a_1]\right)^l \cdot \gamma_m(\beta)\left(v+1-[a_{t-1},\ldots,a_1]\right)^m \ll$$

$$\ll \sum_{v=w}^{\infty} v^k \sum_{a_1+\cdots+a_i=v} \frac{1}{q^{2\beta}} \ll \int_w^{\infty} \frac{dv}{v^{2\beta-k}} = O\left(\frac{1}{w^{2\beta-k-1}}\right).$$

Thus,

$$R_k = D_k + O\left(\frac{1}{w^{2\beta-k-1}}\right),$$

where

$$D_k = \sum_{v=1}^{\infty} \sum_{a_1+\cdots+a_{t-1}=v, a_{t-1}\geq 2} \frac{2}{q_-^{2\beta}} \sum_{l+m=k} \gamma_l(\beta)\left(v - [a_{t-1},\ldots,a_1]\right)^l \cdot \gamma_m(\beta)\left(v+1-[a_{t-1},\ldots,a_1]\right)^m .$$

Now let us consider R_k when $k \geq 2\beta - 1$:

$$R_k \leq 2 \sum_{l+m=k} \gamma_l(\beta) \cdot \gamma_m(\beta) \sum_{v=1}^{w-1} v^k \sum_{a_1+\cdots+a_i=v, a_{t-1}\geq 2} \frac{1}{q^{2\beta}} \leq$$

$$\leq 4\frac{\zeta(2\beta-1)}{\zeta(2\beta)} \sum_{l+m=k} \gamma_l(\beta)\cdot\gamma_m(\beta) \int_1^{w-1} \frac{dv}{v^{2\beta-k}} \leq 4\frac{\zeta(2\beta-1)}{\zeta(2\beta)} \sum_{l+m=k} \gamma_l(\beta)\cdot\gamma_m(\beta) w^{k+1-2\beta},$$

if $k > 2\beta - 1$, and $O(\log w)$ when $k = 2\beta - 1$ (in the case when 2β is integer).
Then for residual series we get the following estimation:

$$\sum_{k>2\beta-1} \frac{R_k}{n^{2\beta+k}} \leq 4\frac{\zeta(2\beta-1)}{\zeta(2\beta)} \sum_{k>2\beta-1} \frac{w^{k+1-2\beta}}{n^{2\beta+k}} \sum_{l+m=k} \gamma_l(\beta) \cdot \gamma_m(\beta) =$$

$$= 4\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\frac{1}{w^{2\beta-1}n^{2\beta}}\sum_{k>2\beta-1}\frac{w^k}{n^k}\sum_{l+m=k}\gamma_l(\beta)\cdot\gamma_m(\beta) \leq$$

$$\leq 4\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\frac{1}{w^{2\beta-1}n^{2\beta}}\sum_{k=1}^{\infty}\frac{w^k}{n^k}\sum_{l+m=k}\gamma_l(\beta)\cdot\gamma_m(\beta) \leq$$

$$\leq 4\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\frac{1}{w^{2\beta-1}n^{2\beta}}\left(\frac{1}{1-\frac{w}{n}}\right)^{2\beta} = O\left(\frac{1}{w^{2\beta-1}n^{2\beta}}\right)$$

with the given w.
Extracting the constant in the main term , we obtain when 2β is integer

$$\Sigma_{n,1}^{(3)+} = \frac{1}{n^{2\beta}}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)} + \sum_{1\leq k<2\beta-1} D_k\frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{\log w}{n^{4\beta-1}} + \frac{1}{n^{(\beta-1)(2r-1)}} + \frac{1}{w^{2\beta-1}n^{2\beta}}\right),$$

when 2β is not integer

$$\Sigma_{n,1}^{(3)+} = \frac{1}{n^{2\beta}}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)} + \sum_{1\leq k<2\beta-1} D_k\frac{1}{n^{2\beta+k}} + O\left(\frac{1}{n^{2\beta}w^{2(\beta-1)}} + \frac{1}{n^{(\beta-1)(2r-1)}} + \frac{1}{w^{2\beta-1}n^{2\beta}}\right).$$

Lemma is proved.
Lemma 15. *When 2β is integer*

$$\Sigma_{n,1}^{(3)-} = \frac{1}{n^{\beta}}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)} + \sum_{1\leq k<2\beta-1} E_k\frac{1}{n^{\beta+k}} + O\left(\frac{\log w}{n^{3\beta-1}} + \frac{1}{n^{\beta}w^{2(\beta-1)}}\right), \quad (28)$$

otherwise

$$\Sigma_{n,1}^{(3)-} = \frac{1}{n^{\beta}}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)} + \sum_{1\leq k<2\beta-1} E_k\frac{1}{n^{\beta+k}} + O\left(\frac{1}{n^{\beta}w^{2(\beta-1)}}\right), \quad (29)$$

where E_k are some constants.
Proof.

$$\Sigma_{n,1}^{(3)-} = \sum_{a\in A_n, q(a)<n^r, a_t>n-w}\frac{1}{\left(qq_-\right)^{\beta}}$$

$$= \sum_{a\in A_n, a_t>n-w}\frac{1}{\left(qq_-\right)^{\beta}} - \sum_{a\in A_n, q(a)\geq n^r}\frac{1}{\left(qq_-\right)^{\beta}}$$

The second sum can be estimated according to lemma 11 as

$$\sum_{a\in A_n, q(a)\geq n^r}\frac{1}{\left(qq_-\right)^{\beta}} = O\left(\frac{1}{n^{(\beta-1)(2r-1)}}\right).$$

For the first sum we have

$$\sum_{a\in A_n, a_t>n-w}\frac{1}{\left(qq_-\right)^{\beta}} = 2\sum_{v=1}^{w}\sum_{\substack{a\in A_n,\\ a_1+\ldots+a_{t-1}=v,\\ a_{t-1}\geq 2}}\frac{1}{\left(qq_-\right)^{\beta}}. \quad (30)$$

Here $a_t = n - v, v = 1, .., w$. As

$$q = a_t q_- + (q_-)_- = nq_- \left(1 - \frac{1}{n}(v - [a_{t-1}, \dots, a_1])\right),$$

then expanding $\frac{1}{(qq_-)^\beta}$ into Teilor series according to parameter $\frac{1}{n}(v-[a_{t-1}, \dots, a_1])$, we get

$$\frac{1}{(qq_-)^\beta} = \frac{1}{n^\beta q_-^{2\beta}} + \frac{1}{n^\beta q_-^{2\beta}} \sum_{k=1}^{\infty} \gamma_k(\beta) \left(\frac{1}{n}(v - [a_{t-1}, \dots, a_1])\right)^k. \qquad (31)$$

Thus, substituting (31) into (30), we obtain

$$\sum_{a \in A_n, a_t > n-w} \frac{1}{(qq_-)^\beta} = 2 \sum_{v=1}^{w} \sum_{\substack{a \in A_n, \\ a_1 + \ldots + a_{t-1} = v, \\ a_{t-1} \geq 2}} \frac{1}{n^\beta q_-^{2\beta}} +$$

$$+ \sum_{k=1}^{\infty} \frac{1}{n^{\beta+k}} 2\gamma_k(\beta) \sum_{v=1}^{w} (v - [a_{t-1}, \dots, a_1])^k \sum_{\substack{a \in A_n, \\ a_1 + \ldots + a_{t-1} = v, \\ a_{t-1} \geq 2}} \frac{1}{q_-^{2\beta}} =$$

$$= 2 \sum_{v=1}^{w} \sum_{\substack{a \in A_n, \\ a_1 + \ldots + a_{t-1} = v, \\ a_{t-1} \geq 2}} \frac{1}{n^\beta q_-^{2\beta}} + \sum_{k=1}^{\infty} \frac{R_k}{n^{\beta+k}},$$

where R_k is defined as follows:

$$R_k = 2\gamma_k(\beta) \sum_{v=1}^{w} \sum_{a \in A_v} \frac{1}{q^{2\beta}} (v - [a_t, \dots, a_1])^k =$$

$$= 2\gamma_k(\beta) \left(\sum_{v=1}^{\infty} - \sum_{v=w}^{\infty}\right) \sum_{a \in A_v} \frac{1}{q^{2\beta}} (v - [a_t, \dots, a_1])^k.$$

Let us consider the first sum:

$$2\gamma_k(\beta) \sum_{v=1}^{\infty} \sum_{a \in A_v} \frac{1}{q^{2\beta}} (v - [a_t, \dots, a_1])^k \ll \sum_{v=1}^{\infty} v^k \sum_{a \in A_v} \frac{1}{q^{2\beta}}.$$

It follows from lemma 9 that $\sum_{a \in A_v} \frac{1}{q^{2\beta}} = O\left(\frac{1}{v^{2\beta}}\right)$, hence

$$\sum_{v=1}^{\infty} v^k \sum_{a \in A_v} \frac{1}{q^{2\beta}} \ll \sum_{v=1}^{\infty} \frac{1}{v^{2\beta-k}}.$$

Thus, the given series converges when $2\beta - k > 1$, i. e. when $k < 2\beta - 1$. With these k let us estimate the residual series of the given series.

$$2\gamma_k(\beta)\sum_{v=w}^{\infty}\sum_{a\in A_v}\frac{1}{q^{2\beta}}\left(v-[a_t,\ldots,a_1]\right)^k \ll$$

$$\ll \sum_{v=w}^{\infty} v^k \sum_{a\in A_v}\frac{1}{q^{2\beta}} \ll \int_w^{\infty}\frac{dv}{v^{2\beta-k}} = O\left(\frac{1}{w^{2\beta-k-1}}\right).$$

Hence, when $k < 2\beta - 1$

$$R_k = E_k + O\left(\frac{1}{w^{2\beta-k-1}}\right).$$

Here E_k are constants, defined with the following formula

$$E_k = 2\gamma_k(\beta)\sum_{v=1}^{\infty}\sum_{a\in A_v}\frac{1}{q^{2\beta}}\left(v-[a_t,\ldots,a_1]\right)^k.$$

Now let's estimate the sum when $k \geq 2\beta - 1$.

$$R_k \leq 4\gamma_k(\beta)C_0\sum_{v=1}^{w-1} v^{k-2\beta} \leq$$

$$\leq 4\gamma_k(\beta)C_0 w^{k+1-2\beta}$$

when $k > 2\beta - 1$ and $O(\log w)$ when $k = 2\beta - 1$. Then, summing according to $k > 2\beta - 1$, we get

$$\sum_{k>2\beta-1}\frac{R_k}{n^{\beta+k}} \leq 4C_0\frac{1}{n^\beta w^{2\beta-1}}\sum_{k>2\beta-1}\gamma_k(\beta)\frac{w^k}{n^k} \leq$$

$$\leq 4C_0\frac{1}{n^\beta w^{2\beta-1}}\left(\frac{1}{1-\frac{w}{n}}\right)^\beta = O\left(\frac{1}{n^\beta w^{2\beta-1}}\right).$$

Extracting the constant in the main term , we get

$$\frac{2}{n^\beta}\sum_{a\in A_n, a_1+\ldots+a_{t-1}\leq w}\frac{1}{q_-^{2\beta}} = \frac{1}{n^\beta}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{1}{n^\beta w^{2(\beta-1)}}\right).$$

Thus, when 2β is integer

$$\Sigma_{n,1}^{(3)-} = \frac{1}{n^\beta}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\leq k<2\beta-1}E_k\frac{1}{n^{\beta+k}} + O\left(\frac{\log n}{n^{3\beta-1}} + \frac{1}{n^\beta w^{2(\beta-1)}}\right),$$

when $2\beta$is not integer

$$\Sigma_{n,1}^{(3)-} = \frac{1}{n^\beta}\frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\leq k<2\beta-1}E_k\frac{1}{n^{\beta+k}} + O\left(\frac{1}{n^\beta w^{2(\beta-1)}}\right).$$

Lemma is proved.

The final step in proving theorem 2. Substituting (14), (15), (17), (17), (24), (24), (28), (29) into (13), we obtain when 2β is integer

$$\Sigma_{n,1}^{(3)-}+\Sigma_{n,1}^{(3)+}+\Sigma_{n,2}^{(3)}+\Sigma_{n,2}^{(2)}+\Sigma_{n,2}^{(1)}=$$

$$=\frac{1}{n^{\beta}}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)}+\sum_{1\leq k<2\beta-2}C_k\frac{1}{n^{\beta+k}}++\sum_{0\leq k<\beta-2}C^*{}_k\frac{1}{n^{2\beta+k}}+$$

$$+O\left(\frac{\log n}{n^{3\beta-2}}+\frac{1}{n^{2\beta}w^{\beta-2}}+\frac{n^2\left(\log^{3\beta}n\right)^{3\beta}}{w^{3\beta}}+\frac{1}{n^{\beta}w^{2(\beta-1)}}+\frac{1}{n^{(\beta-1)(2r-1)}}\right),$$

where $C_k=E_k$, and $C_k^*=B_k+D_k$, $C^*{}_0=B_k+\frac{2\zeta(2\beta-1)}{\zeta(2\beta)}$. When 2β is not integer ,

$$\Sigma_{n,1}^{(3)-}+\Sigma_{n,1}^{(3)+}+\Sigma_{n,2}^{(3)}+\Sigma_{n,2}^{(2)}+\Sigma_{n,2}^{(1)}=$$

$$=\frac{1}{n^{\beta}}\frac{2\zeta\left(2\beta-1\right)}{\zeta\left(2\beta\right)}+\sum_{1\leq k<2\beta-2}C_k\frac{1}{n^{\beta+k}}+\sum_{0\leq k<\beta-2}C^*{}_k\frac{1}{n^{2\beta+k}}+$$

$$+O\left(\frac{1}{n^{2\beta}w^{\beta-2}}+\frac{n^2\left(\log^{3\beta}n\right)^{3\beta}}{w^{3\beta}}+\frac{1}{n^{\beta}w^{2(\beta-1)}}+\frac{1}{n^{(\beta-1)(2r-1)}}\right),$$

where $C_k=E_k$, and $C^*{}_k=B_k+D_k$, $C^*{}_0=B_k+\frac{2\zeta(2\beta-1)}{\zeta(2\beta)}$. To minimize degree of error term, let $w=\frac{n}{2}-2$, $r=\frac{3\beta}{2(\beta-1)}+\frac{1}{2}$.

Theorem is proved.

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